Trigonometric inequality(**1 year**)

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Prove that in any acute-angled triangle ABC the following inequality holds

$\cos^2 A\cos^2 B + \cos^2 B\cos^2 C + \cos^2 C\cos^2 A \le (1/4)(\cos^2 A + \cos^2 B + \cos^2 C)$.

Solution by Arkady Alt , **San Jose**, **California**, **USA**.

Since set of all positive solutions (x,y,z) of equation $x^2+y^2+z^2+2xyz = 1$ can be represented as set $\left\{(\cos A,\cos B,\cos C) \mid A,B,C \in (0,\pi/2)\ \&\ A+B+C = \pi\right\}$ and at the same time as

$$\left\{\left(\sqrt{\frac{yz}{(1-y)(1-z)}},\sqrt{\frac{zx}{(1-z)(1-x)}},\sqrt{\frac{zx}{(1-z)(1-x)}}\right) \mid x,y,z > 0\ \&\ x+y+z = 1\right\}$$

then original inequality is equivalent to inequality

$$\sum\frac{yz}{(1-y)(1-z)}\cdot\frac{zx}{(1-z)(1-x)} \le \frac{1}{4}\sum\frac{yz}{(1-y)(1-z)} \Leftrightarrow \sum\frac{x}{1-x} \le \frac{1}{4}\sum\frac{1-x}{x} \Leftrightarrow$$

$$\sum\frac{1}{1-x} - 3 \le \frac{1}{4}\left(\sum\frac{1}{x} - 3\right) \Leftrightarrow 4\sum\frac{1}{1-x} \le \sum\frac{1}{x} + 9.$$

Let $p := xy+yz+zx, q := xyz$. Then $\sum\frac{1}{1-x} = \frac{1+p}{p-q}, \sum\frac{1}{x} = \frac{p}{q}$ and latter inequality becomes $\frac{4(1+p)}{p-q} \le \frac{p}{q} + 9$. Since* $q \le \frac{p^2}{4-3p}$ and

$3p = 3(xy+yz+zx) \le (x+y+z)^2 = 1$

then $$\frac{p}{q} + 9 - \frac{4(1+p)}{p-q} \ge \frac{p}{\frac{p^2}{4-3p}} - \frac{4(1+p)}{p-\frac{p^2}{4-3p}} + 9 = \frac{4-3p}{p} - \frac{(1+p)(4-3p)}{p(1-p)} + 9 =$$

$$\frac{(4-3p)((1-p)-(1+p)) + 9p(1-p)}{p(1-p)} = \frac{1-3p}{1-p} \ge 0.$$

* $0 \le \sum_{cyc} z^3(x-y)^2 = z^3(x-y)^2 + x^3(y-z)^2 + y^3(z-x)^2 =$

$x^2y^2(x+y) + y^2z^2(y+z) + z^2x^2(z+x) - 2xyz(x^2+y^2+z^2) =$

$(x+y+z)(x^2y^2+x^2z^2+y^2z^2) - xyz(xy+yz+zx) - 2xyz(x^2+y^2+z^2) =$

$(p^2-2q) - qp - 2q(1-2p) = p^2 - q(4-3p)$.